Bounds of efficiency at maximum power for normal-, sub- and super-dissipative Carnot-like heat engines

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The Carnot-like heat engines are classified into three types (normal-, sub- and super-dissipative) according to relations between the minimum irreversible entropy production in the "isothermal" processes and the time for completing those processes. The efficiencies at maximum power of normal-, sub- and super-dissipative Carnot-like heat engines are proved to be bounded between $\eta_C/2$ and $\eta_C/(2-\eta_C)$, $\eta_C/2$ and η_C, 0 and $\eta_C/(2-\eta_C)$, respectively. These bounds are also shared by linear, sub- and super-linear irreversible Carnot-like engines [Tu and Wang, arXiv:1110.6493] although the dissipative engines and the irreversible ones are inequivalent to each other.

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I. INTRODUCTION

The issue of efficiency at the maximum power (EMP) has been drawn much attention since the pioneer achievements make by Yvon[1], Novilov[2], Chambadal[3], Curzon and Ahlborn[4]. The emerging theoretical advances in this field [5–28] improve our understanding of the issue of EMP for heat engines and irreversible thermodynamics. Recently, Esposito *et al.* investigated the Carnot-like engines working in the low-dissipation region [27] and obtained the lower bound $\eta_- \equiv \eta_C/2$ and the upper bound $\eta_+ \equiv \eta_C/(2-\eta_C)$ of EMP for this kind of engines, where η_C is the Carnot efficiency. In addition, Gaveau and his coworkers proposed a concept of sustainable efficiency and proved that it has the upper bound 1/2, based on which they also obtained the upper bound $\eta_+ = \eta_C/(2-\eta_C)$ for the efficiency of Carnot-like engines at maximum power [28]. Seifert argued that the upper bound 1/2 for the sustainable efficiency holds only in the linear nonequilibrium region [29]. Esposito *et al.* [30] provided a nice example of single level quantum dot where the upper bound can vary from 1/2 to 1 with increasing the thermodynamic force, which to some extent supports Seifert's argument. Therefore the upper bound $\eta_C/(2-\eta_C)$ might not exist for EMP of Carnot-like heat engines arbitrarily far from equilibrium [31]. In recent work, we classified irreversible Carnot-like heat engines into three types (linear, sublinear and superlinear), and derived the corresponding bounds to be between $\eta_C/2$ and $\eta_C/(2-\eta_C)$, $\eta_C/2$ and η_C, 0 and $\eta_C/(2-\eta_C)$, respectively [32]. In particular, it is found that the EMP of sublinear irreversible heat engines can reach the Carnot efficiency [32].

It is necessary to introduce the concept of effective temperature of working substance for most of models mentioned above. However, the definition of effective temperature is debatable in some cases. Esposito *et al.* started from the time-dependent behavior of the irreversible entropy production in the "isothermal" (The quote marks on the word "isothermal" merely indicate the working substance to be in contact with a reservoir) process without introducing the effective temperature of working substance [27], which inspired us that the upper and lower bounds of EMP might be straightly derived from the relation between the irreversible entropy production in the "isothermal" process and the time for completing that process. Here, we classify the Carnot-like engines into three types (normal-, sub- and super-dissipative) according to the relations between the minimum irreversible entropy production in the finite-time "isothermal" processes and the time for completing those processes. The EMPs of normal-, sub- and super-dissipative Carnot-like heat engines are proved to be bounded between $\eta_C/2$ and $\eta_C/(2-\eta_C)$, $\eta_C/2$ and η_C, 0 and $\eta_C/(2-\eta_C)$, respectively.

II. THEORETICAL MODEL

The engines perform the following Carnot-like cycle which is similar to the convention proposed by Schmiedl and Seifert [12].

"Isothermal" expansion. The working substance is in contact with a hot reservoir at temperature T_1 and the constraint on the system is loosened according to some external controlled parameter $\lambda_1(\tau)$ during the time interval $0 < \tau < t_1$ where τ is the time variable. It is in the sense of loosening the constraint that this step is called expansion process. A certain amount of heat Q_1 is absorbed from the hot reservoir. Then the variation of entropy can be expressed as

$$\Delta S_1 = Q_1/T_1 + \Delta S_1^{ir}, \tag{1}$$

where $\Delta S_1^{ir} \geq 0$ is the irreversible entropy production.

Adiabatic expansion. The adiabatic expansion is idealized as the working substance suddenly decouples from the hot reservoir and then comes into contact with the cold reservoir instantly at time $\tau = t_1$. During this transition, the constraint on the system is loosened further.

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There is no heat exchange and entropy production in this process, i.e., $Q_2 = 0$ and $\Delta S_2 = 0$.

"Isothermal" compression. The working substance is in contact with a cold reservoir at temperature T_3 and the constraint on the system is enhanced according to the external controlled parameter $\lambda_3(\tau)$ during the time interval $t_1 < \tau < t_1 + t_3$. It is in the sense of enhancing the constraint that this step is called compression process. A certain amount of heat Q_3 is released to the cold reservoir. The variation of entropy in this process can be expressed as

$$\Delta S_3 = -Q_3/T_3 + \Delta S_3^{ir}, \tag{2}$$

where $\Delta S_3^{ir} \geq 0$ is the irreversible entropy production.

Adiabatic compression. Similar to the adiabatic expansion, the working substance suddenly decouples from the cold reservoir and then comes into contact with the hot reservoir instantaneously at time $\tau = t_1 + t_3$. The constraint on the system is enhanced further. There is no heat exchange and entropy production in this process, i.e., $Q_4 = 0$ and $\Delta S_4 = 0$.

Having undergone this Carnot-like cycle, the system comes back to its initial state again. Thus there are no net energy change and variation of entropy in the whole cycle. Then we have $\Delta S_3 = -\Delta S_1$ and and the net work output $W = Q_1 - Q_3$. The power can be expressed as

$$P = \frac{Q_1 - Q_3}{t_{tot}} = \frac{(T_1 - T_3)\Delta S_1 - T_1\Delta S_1^{ir} - T_3\Delta S_3^{ir}}{t_1 + t_3}. \tag{3}$$

According to the equation above, maximizing the power means minimizing the irreversible entropy production with respect to the protocols $\lambda_1(\tau)$ and $\lambda_1(\tau)$ for given time intervals t_1 and t_3 first, then Eq. (3) can be transformed into

$$P = \frac{(T_1 - T_3)\Delta S_1 - T_1 L_1 - T_3 L_3}{t_1 + t_3}, \tag{4}$$

where L_1 and L_3 represent $\min\{\Delta S_1^{ir}\}$ and $\min\{\Delta S_1^{ir}\}$ for given time intervals t_1 and t_3, respectively. In fact, they reflect to what extent the engines depart from equilibrium for given time intervals t_1 and t_3.

III. CLASSIFICATION FOR THREE TYPES OF DISSIPATIVE HEAT ENGINES

Intuitionally, the irreversible entropy production decreases as the increase of time for completing the "isothermal" processes, thus L_1 and L_3 can be expressed as monotone increasing function with respect to $1/t_1$ and $1/t_3$, respectively. If we introduce a transformation of variables $x_i = 1/t_i$ $(i = 1, 3)$, the minimum irreversible entropy production in each "isothermal" process can be expressed as $L_i = L_i(x_i)$ $(i = 1, 3)$. We can also defined three kinds of typical characteristics according to the behavior of $L_i = L_i(x_i)$, which is similar to the analysis in our previous work [32]. The first one is called normal dissipative type which is represented by the straight line in Fig. 1. This type of Carnot-like engines is also called the low-dissipation engines by Esposito *et al.* [27]. The second one is called sub-dissipative type which is represented by the convex curve in Fig.1. The third one is called super-dissipative type which is represented by the concave curve in Fig.1. The behavior of three kinds of characteristics can be mathematically expressed as

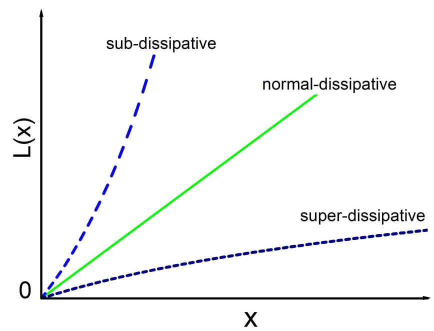

FIG. 1. Schematic diagram of three types of dissipative engines.

$$\begin{cases} xL' = L, & \text{normal-dissipative} \\ xL' > L, & \text{sub-dissipative} \\ xL' < L, & \text{super-dissipative} \end{cases} \tag{5}$$

where L, x, and L' represent L_1(or L_3), x_1(or x_3) and $\mathrm{d}L_1/\mathrm{d}x_1$ (or $\mathrm{d}L_3/\mathrm{d}x_3$), respectively.

IV. OPTIMIZATION

The heat absorbed or released by the engines can be expressed as

$$Q_1 = T_1\Delta S_1 - T_1 L_1, \tag{6}$$

and

$$Q_3 = T_3\Delta S_1 + T_3 L_3. \tag{7}$$

Given the two equations above, we can obtain the efficiency

$$\eta = 1 - \frac{Q_3}{Q_1} = \frac{\eta_C\Delta S_1 - L_1 - (1 - \eta_C)L_3}{\Delta S_1 - L_1}, \tag{8}$$

and the power

$$P = \frac{[(T_1 - T_3)\Delta S_1 - (T_1 L_1 + T_3 L_3)]x_1 x_3}{(x_1 + x_3)}. \tag{9}$$

Maximizing the power with respect to x_1 and x_3, we can obtain

$$[(T_1 - T_3)\Delta S_1 - (T_1 L_1 + T_3 L_3)]x_3 = T_1 x_1 (x_1 + x_3) L_1', \tag{10}$$

and

$$[(T_1 - T_3)\Delta S_1 - (T_1 L_1 + T_3 L_3)]x_1 = T_3 x_3 (x_1 + x_3) L_3'. \tag{11}$$

Dividing Eqs. (10) and (11), we derive

$$T_1 x_1^2 L_1' = T_3 x_3^2 L_3'. \tag{12}$$

Adding Eqs. (10) and (11) with the consideration of Eq. (8), we find that the EMP satisfies

$$\eta^* = \frac{\eta_C}{1 + \frac{(1-\eta_C)(L_1+L_3)}{x_1 L_1' + (1-\eta_C) x_3 L_3'}}, \tag{13}$$

which is the key equation in our paper.

V. BOUNDS OF EMP FOR THREE TYPES OF DISSIPATIVE HEAT ENGINES

In this section, we will discuss the bounds of EMP for three types of Carnot-like dissipative engines in terms of the characteristics of the relation between the minimum entropy production in each "isothermal" processes and the time for completing those processes.

A. Normal-dissipative engines

The minimum entropy production in the "isothermal" processes and the time for completing those processes of normal dissipative engines satisfy $x_1 L_1' = L_1$ and $x_3 L_3' = L_3$ which imply $x_1 L_1' + (1-\eta_C) x_3 L_3' = L_1 + (1-\eta_C) L_3$. Considering $0 < \eta_C < 1$, we derive $(1-\eta_C)(L_1 + L_3) < L_1 + (1-\eta_C)L_3 < L_1 + L_3$. Considering Eq. (13), we can derive the EMP of normal-dissipative engines to be bounded between $\eta_- \equiv \eta_C/2$ and $\eta_+ \equiv \eta_C/(2-\eta_C)$ which are the same as the bounds obtained by Esposito and his coworkers [27]. A major difference is that here we derive the bounds directly from Eq. (13) without calculating the explicit expression of EMP.

B. Sub-dissipative engines

The sub-dissipative engines satisfy $x_1 L_1' > L_1$ and $x_3 L_3' > L_3$ which imply $x_1 L_1' + (1-\eta_C) x_3 L_3' > L_1 + (1-\eta_C) L_3$. Given that $L_1 + (1-\eta_C)L_3 > (1-\eta_C)(L_1 + L_3)$, then we finally derive the lower bound of EMP to be $\eta_- = \eta_C/2$ for the sub-dissipative engines from Eq. (13). The above inequalities give no confinement on the upper bound, thus we may take $\eta_+ = \eta_C$ as a reasonable estimate.

C. Super-dissipative engines

The super-dissipative engines satisfy $x_1 L_1' < L_1$ and $x_3 L_3' < L_3$ which imply $x_1 L_1' + (1-\eta_C) x_3 L_3' < L_1 + (1-\eta_C) L_3$. Given that $L_1 + (1-\eta_C)L_3 < L_1 + L_3$, then we finally derive the upper bound of EMP to be $\eta_+ = \eta_C/(2-\eta_C)$ for the super-dissipative engines. The above inequalities give no confinement on the lower bound, thus we may take $\eta_- = 0$ as a conservative estimate.

D. Examples for three types of engines

For examples, we consider the relation of power-law profile, $L_i = \Gamma_i x_i^n, (i = 1, 3)$ where $\Gamma_i > 0$ and $n > 0$ are given parameters. It is easy to see that a heat engine is of normal-, sub- or super-dissipative type if $n = 1$, $n > 1$ or $0 < n < 1$, respectively. Substituting this relation into Eqs. (12) and (13), we can explicitly derive the expression of EMP as

$$\eta^* = \frac{\eta_C}{1 + \frac{1}{n} - \frac{\eta_C}{n[1 + (T_3\Gamma_3/T_1\Gamma_1)^{1/(n+1)}]}}, \tag{14}$$

from which we can find

$$\frac{\eta_C}{1 + 1/n} < \eta^* < \frac{\eta_C}{1 + 1/n - \eta_C/n}. \tag{15}$$

For normal-dissipative engines, $n = 1$, Eq. (15) implies that $\eta_C/2 < \eta^* < \eta_C/(2-\eta_C)$. For sub-dissipative engines, $n > 1$, Eq. (15) implies that $\eta_C/2 < n\eta_C/(n+1) < \eta^* < n\eta_C/(n+1-\eta_C) < \eta_C$ where the upper bound η_C can be reached for sufficiently large n while the lower bound can be reached for $n \to 1_+$. For super-dissipative engines, $0 < n < 1$, Eq. (15) implies that $0 < n\eta_C/(n+1) < \eta^* < n\eta_C/(n+1-\eta_C) < \eta_C/(2-\eta_C)$ where the lower bound 0 can be reached for small enough n while the upper bound can be reached for $n \to 1_-$.

VI. CONCLUSION

The heat engines are classified into three dissipative types according to characteristics of the relations between the minimum irreversible entropy production in the "isothermal" processes and the time for completing those processes. The bounds of EMP for three types of dissipative Carnot-like engines can be summarized as

$$\begin{cases} \eta_C/2 < \eta^* < \eta_C/(2-\eta_C), & \text{normal}-\text{dissipative} \\ \eta_C/2 < \eta^* < \eta_C, & \text{sub}-\text{dissipative} \\ 0 < \eta^* < \eta_C/(2-\eta_C), & \text{super}-\text{dissipative} \end{cases}$$

which display certain universality for each types of dissipative Carnot-like engines. It is interesting that the bounds of EMPs for normal-, sub- and super-dissipative Carnot-like engines correspond respectively to those for

linear, sub- and super-linear irreversible Carnot-like engines discussed in our previous work [32]. However, a simple consideration implies that these two kinds of classifications are different from each other [24]. It is still an open question to understand why they share the same bounds.

VII. ACKNOWLEDGEMENT

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